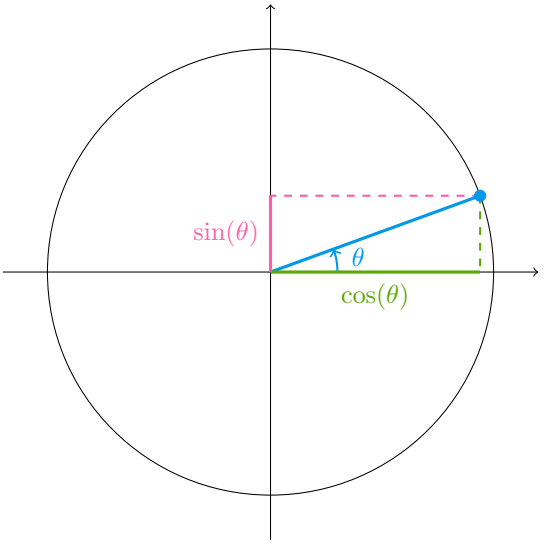
Fiche non-exhaustive de trigonométrie circulaire

Cosinus et sinus

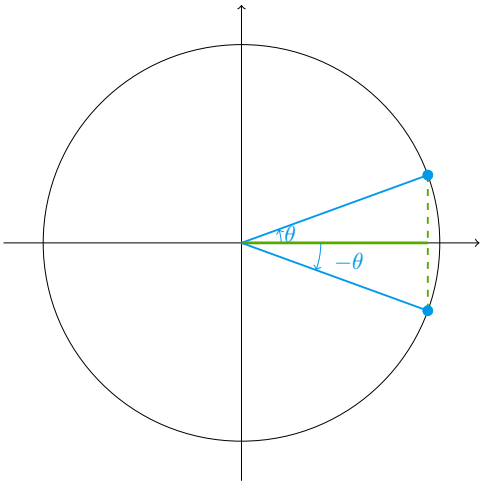
$$\forall \theta \in \mathbb{R}, \boxed{e^{i\theta} = \cos(\theta) + i\,\sin(\theta)}$$



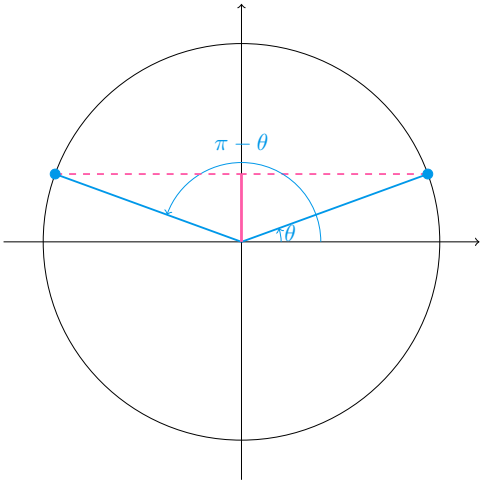
Équations trigonométriques

Pour tous $\theta_1, \theta_2 \in \mathbb{R}$,

$$\cos(\theta_1) = \cos(\theta_2) \Leftrightarrow \begin{cases} \theta_1 \equiv \theta_2 \ [2\pi] \\ \text{ou} \\ \theta_1 \equiv -\theta_2 \ [2\pi] \end{cases}$$

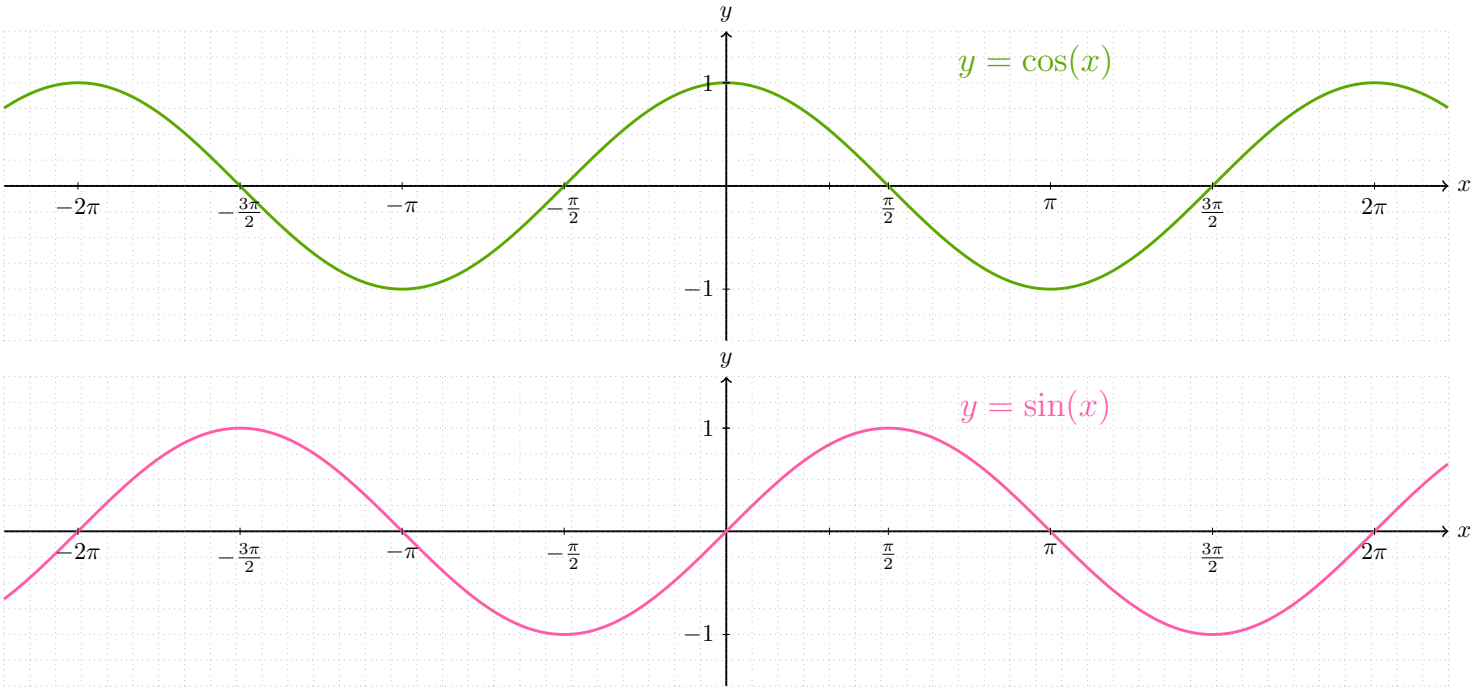


$$\sin(\theta_1) = \sin(\theta_2) \Leftrightarrow \begin{cases} \theta_1 \equiv \theta_2 \ [2\pi] \\ \text{ou} \\ \theta_1 \equiv \pi - \theta_2 \ [2\pi] \end{cases}$$



Fonctions cosinus et sinus

La fonction cos est paire et 2π-périodique.
La fonction sin est impaire et 2π-périodique.



La dérivée de cos est $-\sin$.

$$\forall x \in \mathbb{R},\ \cos'(x) = -\sin(x)$$

Les primitives de cos sont les sin +cste

$$\forall (a,b) \in \mathbb{R}^2, \int_a^b \cos(x)dx = [\sin(x)]_a^b$$

La dérivée de sin est cos.

$$\forall x \in \mathbb{R},\ \sin'(x) = \cos(x)$$

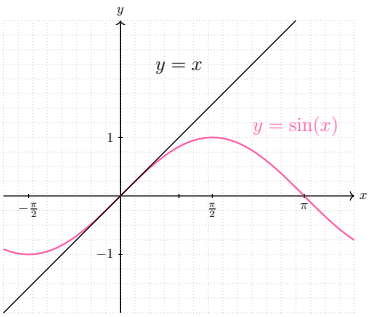
Les primitives de sin sont les $-\cos$ +cste

$$\forall (a,b) \in \mathbb{R}^2, \int_a^b \sin(x)dx = [-\cos(x)]_a^b$$

Comparaison de $\sin(x)$ et x

$$\forall x \in \mathbb{R},\ |\sin(x)| \leq |x|$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$



Oscillateur harmonique

Soit $\omega > 0$. La solution de l'équation différentielle

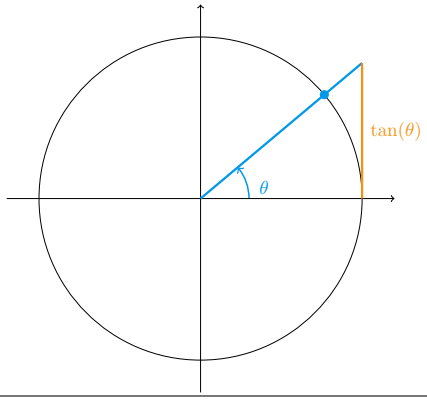
$$\forall t \in \mathbb{R},\ x''(t) + \omega^2 x(t) = 0$$

muni des conditions initiales $x(0) = x_0$ et $x'(0) = v_0$ est

$$t \mapsto x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

Tangente

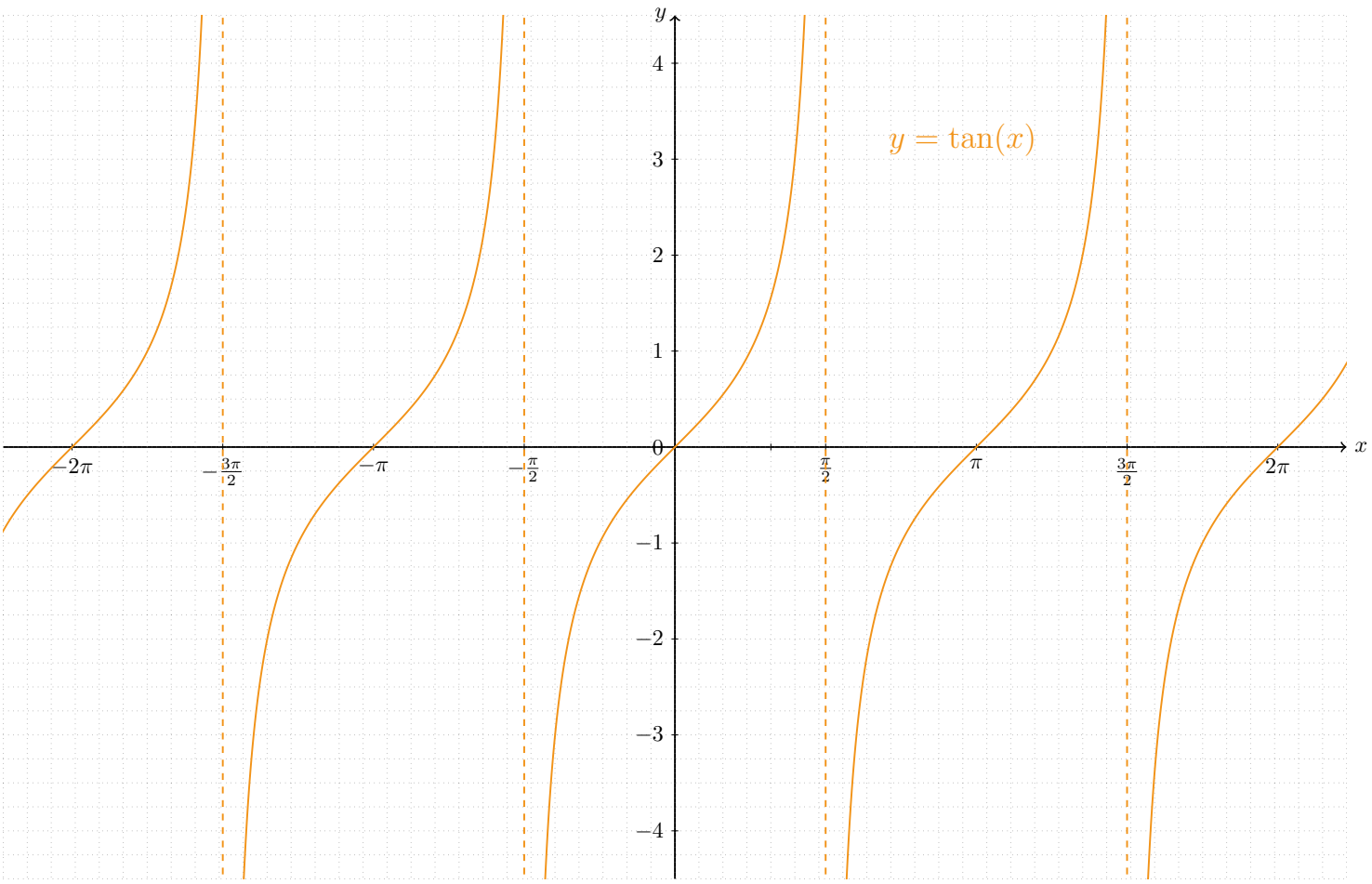
$$\forall \theta \in \mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}, \boxed{\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}}$$



Pour tous $\theta_1, \theta_2 \in \mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}$

$$(\tan(\theta_1)=\tan(\theta_2)) \Leftrightarrow (\theta_1 \equiv \theta_2[\pi]).$$

La fonction tan est impaire et π-périodique.



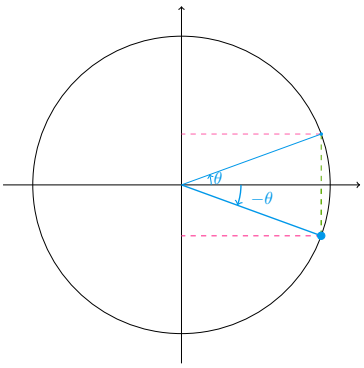
$$\forall x \in \mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}, \quad \tan'(x)=1+\tan^2(x)=\frac{1}{\cos^2(x)}$$

$$\forall x \in \mathbb{R}\backslash\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}, \quad \tan(x)=-\frac{-\sin(x)}{\cos(x)}=-\frac{\cos'(x)}{\cos(x)}$$

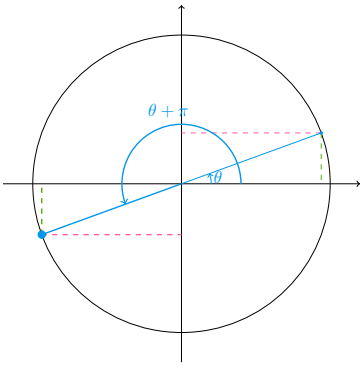
Relations remarquables

Pour tout $\theta \in \mathbb{R}$,

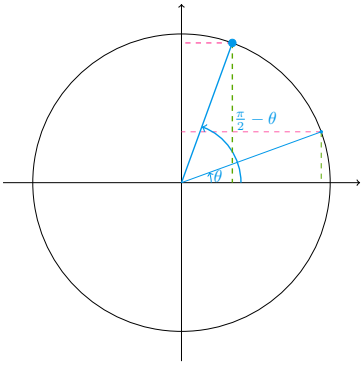
$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$



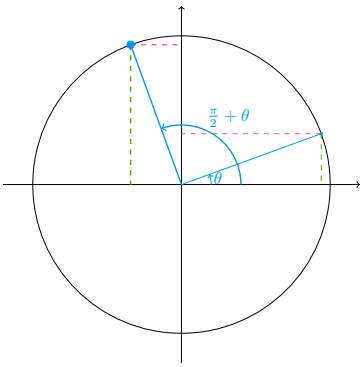
$$\cos(\theta + \pi) = \cos(\theta - \pi) = -\cos(\theta)$$
$$\sin(\theta + \pi) = \sin(\theta - \pi) = -\sin(\theta)$$
$$\tan(\theta + \pi) = \tan(\theta)$$



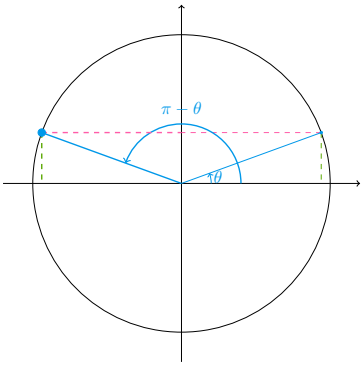
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan(\theta)}$$



$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$$
$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos(\theta)$$
$$\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{1}{\tan(\theta)}$$

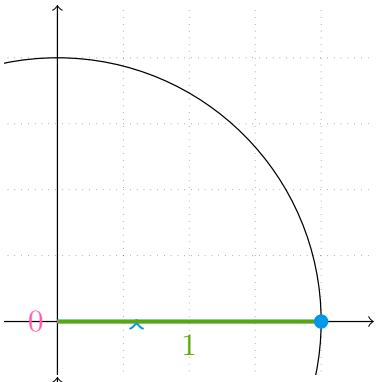


$$\cos(\pi - \theta) = -\cos(\theta)$$
$$\sin(\pi - \theta) = \sin(\theta)$$
$$\tan(\pi - \theta) = -\tan(\theta)$$

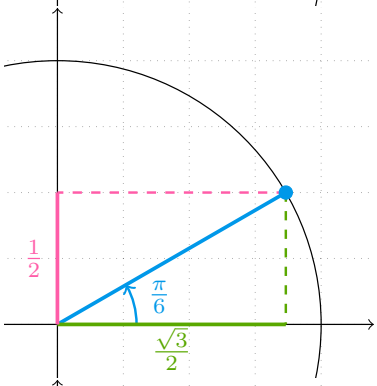


Valeurs remarquables

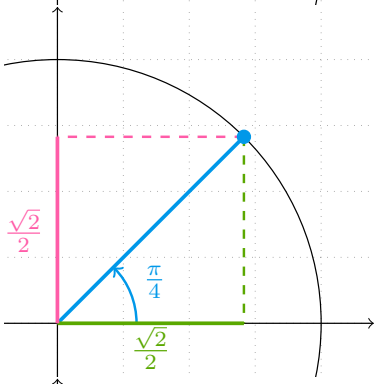
$$\cos(0) = 1$$
$$\sin(0) = 0$$
$$\tan(0) = 0$$



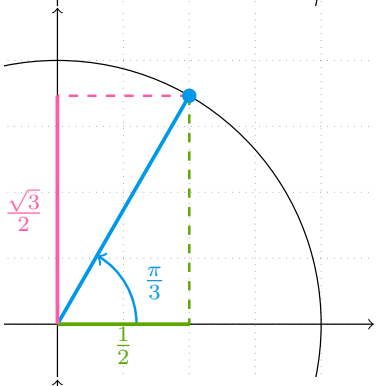
$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$
$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$



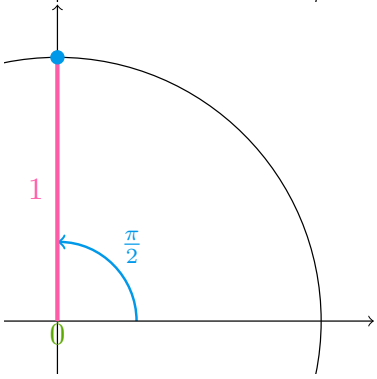
$$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$
$$\tan\left(\frac{\pi}{4}\right) = 1$$



$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$
$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$



$$\cos\left(\frac{\pi}{2}\right) = 0$$
$$\sin\left(\frac{\pi}{2}\right) = 1$$



$$\frac{\sqrt{0}}{2} < \frac{\sqrt{1}}{2} < \frac{\sqrt{2}}{2} < \frac{\sqrt{3}}{2} < \frac{\sqrt{4}}{2}$$

Sommes essentielles

$$\cos(x)^2 + \sin(x)^2 = 1$$

Formules de somme

$$\begin{aligned}
\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\
\cos(a-b) &= \cos(a)\cos(b) + \sin(a)\sin(b) \\
\sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b) \\
\sin(a-b) &= \sin(a)\cos(b) - \cos(a)\sin(b)
\end{aligned}$$

$$\begin{aligned}
\tan(a+b) &= \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)} \\
\tan(a-b) &= \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}
\end{aligned}$$

Duplication

$$\begin{aligned}
\cos(2x) &= \cos(x)^2 - \sin(x)^2 \\
&= 2\cos(x)^2 - 1 \\
&= 1 - 2\sin(x)^2 \\
\sin(2x) &= 2\sin(x)\cos(x) \\
\tan(2x) &= \frac{2\tan(x)}{1 - \tan(x)^2}
\end{aligned}$$

Somme de cosinus

$$\begin{aligned}
\cos(p) + \cos(q) &= \cos\left(\frac{p+q}{2} + \frac{p-q}{2}\right) + \cos\left(\frac{p+q}{2} - \frac{p-q}{2}\right) \\
&= 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)
\end{aligned}$$

ou encore

$$\cos(p) + \cos(q) = \mathrm{Re}\left(e^{ip} + e^{iq}\right) \ = 2\cos\left(\frac{p-q}{2}\right)\mathrm{Re}\left(e^{i\frac{p+q}{2}}\right)$$

De même pour $\cos(p) - \cos(q)$, $\sin(p) + \sin(q)$, $\sin(p) - \sin(q)$.

Cas particuliers

$$\cos(x + n\pi) = (-1)^n \cos(x) \qquad \sin(x + n\pi) \ = (-1)^n \sin(x)$$

Formules d'Euler

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta)$$

$$\cos(\theta) = \mathrm{Re}(e^{i\theta}) \qquad \sin(\theta) = \mathrm{Im}(e^{i\theta})$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Formule amplitude/phase

$$\alpha\cos(\theta) + \beta\sin(\theta) = \sqrt{\alpha^2+\beta^2}\cos(\theta - \varphi)$$

avec φ vérifiant

$$\cos(\varphi) = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \quad \text{et} \quad \sin(\varphi) = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}.$$

Linéarisation

Petites linéarisations

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

$$\cos^3(\theta) = \frac{3\cos(\theta)+\cos(3\theta)}{4} \qquad \sin^3(\theta) = \frac{3\sin(\theta)-\sin(3\theta)}{4}$$

Produits de deux termes

$$\begin{aligned} 2\cos(a)\cos(b) &= \cos(a-b) + \cos(a+b) \\ 2\sin(a)\cos(b) &= \sin(a-b) + \sin(a+b) \\ 2\sin(a)\sin(b) &= \cos(a-b) - \cos(a+b) \end{aligned}$$

Application : puissance moyenne

$$\frac{1}{\left(\frac{2\pi}{\omega}\right)}\int_0^{\frac{2\pi}{\omega}} \cos(\omega t)\cos(\omega t+\varphi)dt = \frac{1}{2}\cos(\varphi).$$

Méthode

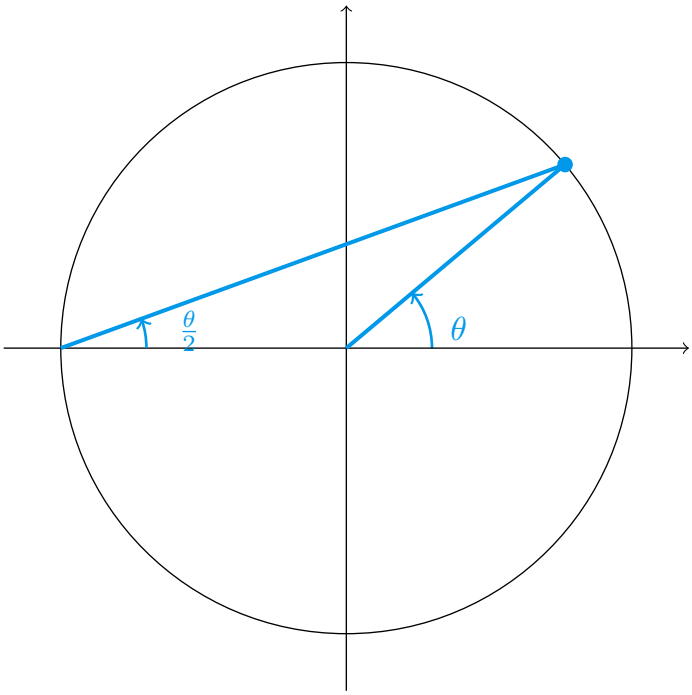
$$\begin{aligned} \cos(\theta)^4 &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 \\ &= \frac{1}{2\times 8}\left(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}\right) \\ &= \frac{1}{8}\left(3 + 4\cos(2\theta) + \cos(4\theta)\right) \end{aligned}$$

Angle moitié

Somme et angle moitié

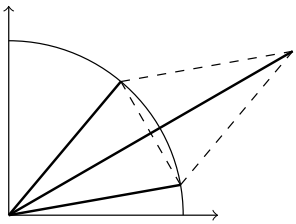
$$\forall \theta \in \mathbb{R}, e^{i\theta} - 1 = e^{i\frac{\theta}{2}} 2i \sin\left(\frac{\theta}{2}\right)$$

$$\forall \theta \in \mathbb{R}, e^{i\theta} + 1 = e^{i\frac{\theta}{2}} 2 \cos\left(\frac{\theta}{2}\right)$$



Pour tous $\theta, \varphi \in \mathbb{R}$,

$$e^{i\theta} + e^{i\varphi} = 2\cos\left(\frac{\theta - \varphi}{2}\right) e^{i\frac{\theta+\varphi}{2}}$$



Tangente de l'angle moitié

Soit $\theta \in \mathbb{R} \backslash \{\pi + 2k\pi, k \in \mathbb{Z}\}$. Avec

$$t = \tan\left(\frac{\theta}{2}\right)$$

on a

$$\cos(\theta) = \frac{1 - t^2}{1 + t^2} \qquad \sin(\theta) = \frac{2t}{1 + t^2} \qquad \tan(\theta) = \frac{2t}{1 - t^2}$$

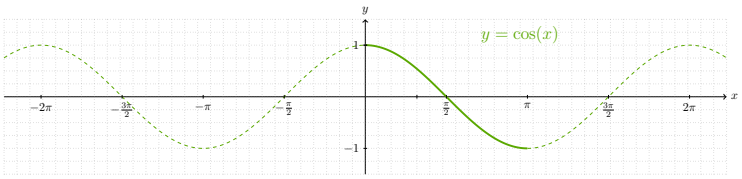
Arctangente et angle moitié

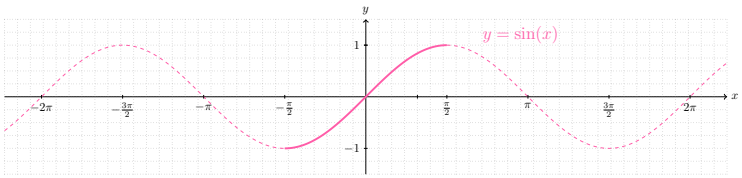
Pour $z = x + iy \in \mathbb{C} \backslash \mathbb{R}_-$, un argument de z est $2\arctan\left(\dfrac{y}{x + \sqrt{x^2 + y^2}}\right)$.

Autres angles moitiés

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 + \cos(x)}{2}} \qquad \sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos(x)}{2}}$$

Fonctions trigonométriques réciproques : définition



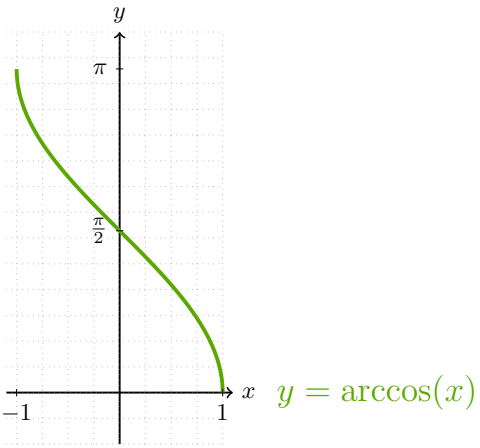


$$\forall \theta \in [0, \pi], \forall x \in [-1, 1]$$

$$(\cos(\theta) = x) \Leftrightarrow (\theta = \arccos(x))$$

$$\forall x \in [-1, 1],\ \cos(\arccos(x)) = x$$

$$\forall \theta \in [0, \pi],\ \arccos(\cos(\theta)) = \theta$$

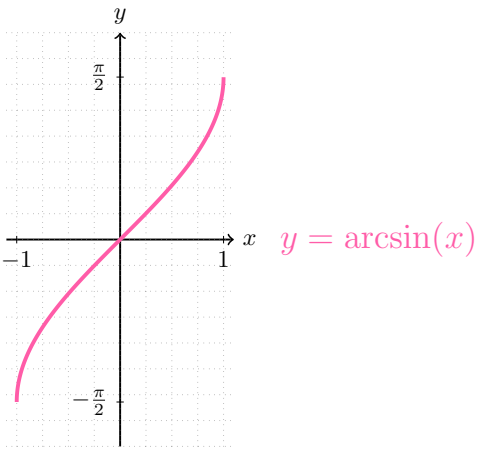


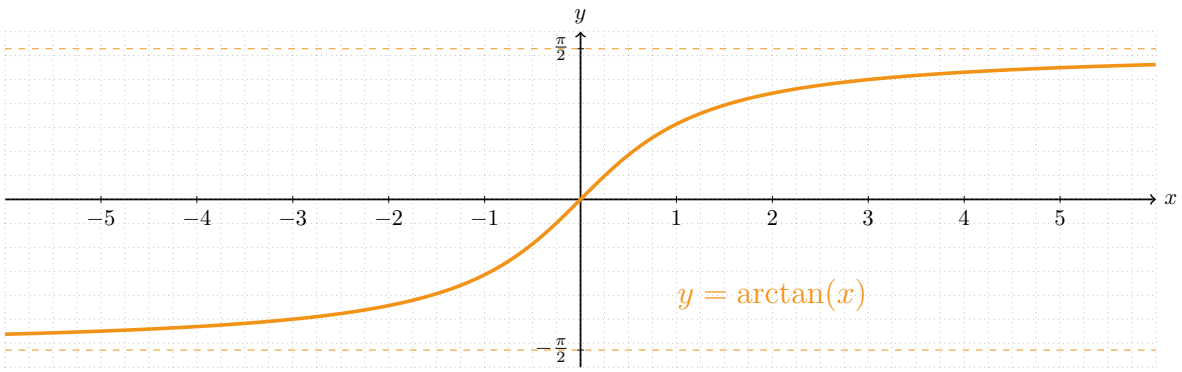
$$\forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \forall x \in [-1, 1]$$

$$(\sin(\theta) = x) \Leftrightarrow (\theta = \arcsin(x))$$

$$\forall x \in [-1, 1],\ \sin(\arcsin(x)) = x$$

$$\forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \arcsin(\sin(\theta)) = \theta$$





$$\forall \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \forall x \in \mathbb{R} \quad (\tan(\theta) = x) \Leftrightarrow (\theta = \arctan(x))$$

$$\forall x \in \mathbb{R},\ \tan(\arctan(x)) = x$$

$$\forall \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[,\ \arctan(\tan(\theta)) = \theta$$

Fonctions trigonométriques réciproques et analyse réelle

Dérivées et primitives de arccos et arcsin

$$\forall x \in]-1,1[,\ \arccos'(x) = \frac{-1}{\sqrt{1-x^2}} \qquad \forall x \in]-1,1[,\ \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$$

Pour $(a,b) \in [-1,1]$, par IPP

$$\int_a^b 1 \cdot \arccos(x)\mathrm{d}x = [x\arccos(x)]_a^b + \int_a^b \frac{2x}{2\sqrt{1-x^2}}\mathrm{d}x$$

$$\int_a^b 1 \cdot \arcsin(x)\mathrm{d}x = [x\arcsin(x)]_a^b - \int_a^b \frac{2x}{2\sqrt{1-x^2}}\mathrm{d}x$$

Relations entre arccos et arcsin

$$\forall x \in [-1,1],\ \arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

$$\forall x \in [-1,1],\ \cos(\arcsin(x)) = \sqrt{1-x^2} \qquad \forall x \in [-1,1],\ \sin(\arccos(x)) = \sqrt{1-x^2}$$

Utilisation de arctan

$$\forall x \in \mathbb{R},\ \arctan'(x) = \frac{1}{1+x^2}$$

$$\forall (a,b) \in \mathbb{R}^2,\ \int_a^b \frac{1}{x^2+1}\mathrm{d}x = [\arctan(x)]_a^b$$

$$\forall \delta > 0,\ \forall (a,b) \in \mathbb{R}^2,\ \int_a^b \frac{1}{x^2+\delta^2}\mathrm{d}x = \left[\frac{1}{\delta}\arctan\left(\frac{x}{\delta}\right)\right]_a^b$$

Pour $(a,b) \in \mathbb{R}^2$, par IPP

$$\int_a^b 1 \cdot \arctan(x)\mathrm{d}x = [x\arctan(x)]_a^b - \frac{1}{2}\int_a^b \frac{2x}{1+x^2}\mathrm{d}x$$

$$\begin{aligned} \forall x > 0,\ \arctan(x) + \arctan\left(\frac{1}{x}\right) &= \frac{\pi}{2} \\ \forall x < 0,\ \arctan(x) + \arctan\left(\frac{1}{x}\right) &= -\frac{\pi}{2} \end{aligned}$$

Addition d'arctangentes

$$\arctan(x) + \arctan(y) - \arctan\left(\frac{x+y}{1-xy}\right) = \begin{cases} 0 & \text{si } xy < 1 \\ \pi & \text{si } xy > 1 \text{ et } x,y > 0 \\ -\pi & \text{si } xy > 1 \text{ et } x,y < 0 \end{cases}$$

Divers

Polynômes de Tchebychev

Relation de récurrence :

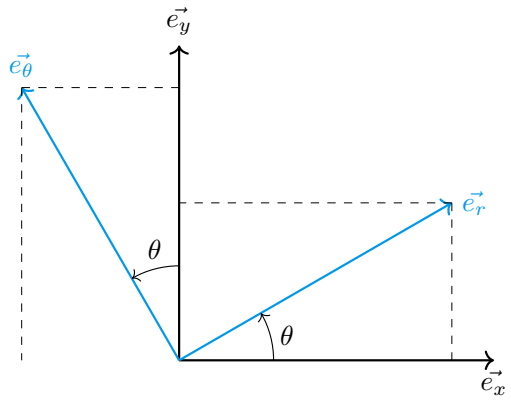
$$\begin{aligned} T_0(X) &= 1 \\ T_1(X) &= X \\ \forall n \in \mathbb{N},\quad T_{n+2}(X) &= 2XT_{n+1}(X) - T_n(X) \end{aligned}$$

Pour tout $n \in \mathbb{N}$,

$$\forall \theta \in \mathbb{R},\ \cos(n\theta) = T_n(\cos(\theta))$$

Changement de repère

$$\begin{aligned} \vec{e_r} &= +\cos(\theta)\vec{e_x} + \sin(\theta)\vec{e_y} \\ \vec{e_\theta} &= -\sin(\theta)\vec{e_x} + \cos(\theta)\vec{e_y} \end{aligned}$$



Interprétation matricielle

On a pour tous $(\theta, \varphi) \in \mathbb{R}^2$,

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix}$$

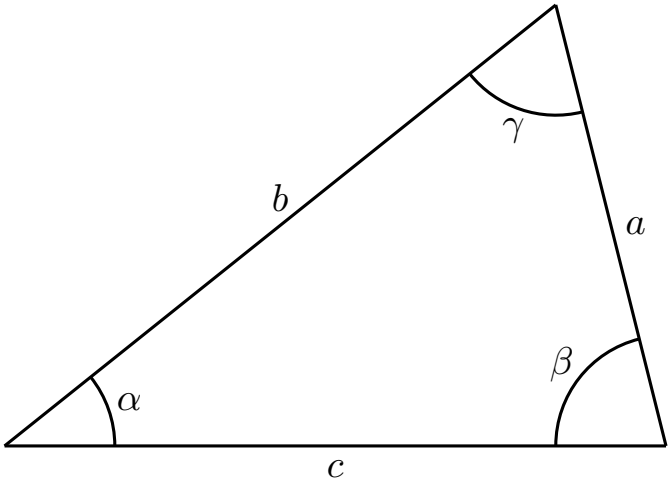
Géométrie

Formule d'Al-Kashi

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

Loi des sinus

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$



Votre propre cercle trigonométrique

